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Note

Matrix Generator of Pseudorandom Numbers

In the work [1] it was proposed to use the Komogorov's K-systems as the pseudorandom numbers generators for the Monte Carlo simulation of physical processes. The main idea of this approach is as follows: The pseudorandom series $\{P_N\}$ is represented by the trajectory of a particular unstable dynamical system, and the phase space is a Π^d-hypercube in the space of dimension d. The system must be maximally unstable in order for the trajectory to fill this hypercube Π^d uniformly. As it is well known, these are the Kolmogorov's K-systems.

In [1] it was proposed to use for that the automorphisms of compact commutative groups, which are defined by the integer matrix $A = \|a_{ij}\|$

$$P_N = AP_{N-1} \bmod 1, \tag{1}$$

where P is the d-dimensional vector, belonging to Π^d, $P_i = (X_1^{(i)}, X_2^{(i)}, ..., X_d^{(i)})$, and

$$\det A = \pm 1 \tag{2}$$

in order to preserve phase space volume Π^d.

To ensure the automorphism (1) to be a K-system, all its eigenvalues must satisfy the condition:

$$|\lambda_K| \neq 1, \qquad K = 1, ..., d \tag{3}$$

Thus, the problem of the pseudorandom sequence $\{P_N\}$ construction reduces to the construction of the matrix A which satisfies the conditions (2) and (3).

Here we propose a simple recurrent way for construction of such a matrix A of arbitrary order d. Suppose we have a matrix of the dimension $\|d-1 \otimes d-1\|$ with $\det A = 1$. Construct now a new matrix the type $\|d \otimes d\|$:

$$A = \begin{Vmatrix} 1 & 0 \quad 0 \quad \cdots \quad 0 \\ 0 & \\ \vdots & \\ & \{\|d-1 \otimes d-1\|\} \\ 0 & \end{Vmatrix} \tag{4}$$

It is evident that $\det A$ is also equal to 1.

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Since the calculation of det A is invariant with respect to the addition of the lines and columns, one can get nontrivial matrices which have different elements in the first line and in the first column, without changing the value of det A. This procedure allows us to get a large set of matrices, which are suitable for the automorphism (1). One needs, however, to check whether all $|\lambda_k|$ are not equal to one. We propose for the matrices A the form:

$$A_d = \begin{Vmatrix} 2 & 3 & \cdots & d & & 1 \\ 1 & 2 & \cdots & d-1 & & 1 \\ 1 & 1 & \cdots & d-2 & & 1 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \\ 1 & 1 & \cdots & 2 & 2 & 1 \\ 1 & 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{Vmatrix}; \tag{5}$$

if $d = 2, 3, 4$, we have

$$A_2 = \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix}, \quad A_3 = \begin{Vmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{Vmatrix}, \quad A_4 = \begin{Vmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{Vmatrix}. \tag{6}$$

We have checked condition (3) for the matrix A_d in the wide range of d ($d = 2, ..., 170$). All cases checked were fulfilled to the condition (3). An example of such calculation for $d = 170$ is given in the Table I. The calculation shows that all $|\lambda_k|$ are in the range $\sim 0.2 \leqslant |\lambda_k| \leqslant 1539.9$ and that the Kolmogorov's entropy

$$h(A) = \sum_{|\lambda_k| > 1} \ln |\lambda_k|$$

increases with d:

$$h(A_{75}) \approx 47.5; \qquad h(A_{110}) \approx 70.1; \qquad h(A_{170}) \approx 108.9.$$

From that we obtain the characteristic time for the decay of correlations [1],

$$\tau \leqslant 1/h(A_d),$$

which decreases with d.

In order to realize the transformation (1) and to construct the concrete trajectory, one must define the initial vector,

$$P_0 = (X_1^{(0)}, ..., X_d^{(0)}), \tag{7}$$

TABLE I

		$\mathrm{Re}\lambda_K$	$\mathrm{Im}\lambda_K$	$\mathrm{Re}\lambda_K$	$\mathrm{Im}\,\lambda_K$
N=	170	-0.58970	0.92975	0.14809	-0.32080
$\mathrm{Re}\lambda_K$	$\mathrm{Im}\lambda_K$	-0.58970	-0.02075	0.14809	-0.32080
1539.88525	0.00000	-0.51913	0.88842	0.15639	0.30852
-177.21851	636.29004	-0.51913	-0.88842	0.15639	-0.30852
-177.21851	-636.29004	-0.45855	0.85218	0.16517	0.29533
-167.06267	125.76009	-0.45855	-0.85128	0.16517	-0.29533
-167.06267	-125.76009	-0.39966	0.81641	0.17201	0.28229
-83.64687	39.28154	-0.39966	-0.81641	0.17201	-0.28229
-83.64687	-39.28154	-0.34836	0.78305	0.17826	0.26946
-47.59344	18.16008	-0.34836	-0.78305	0.17826	-0.26946
-47.59344	-18.16008	-0.30112	0.74994	0.18384	0.25821
-30.22507	10.69967	-0.30112	-0.74994	0.18384	-0.25821
-30.22507	-10.69967	-0.25926	0.71939	0.18947	0.24681
-20.73701	7.32249	-0.25926	-0.71937	0.18947	-0.24681
-20.73701	-7.32249	-0.22253	0.69273	0.19654	0.23554
-15.03566	5.50904	-0.22253	-0.69273	0.19654	-0.23554
15.03566	-5.50904	-0.18580	0.66651	0.20162	0.22382
-11.35427	4.40811	-0.18580	-0.66651	0.20162	-0.22382
-11.35427	-4.40811	-0.15308	0.64134	0.20611	0.21215
-8.84335	3.67899	-0.15308	-0.64164	0.20611	-0.21215
-8.84335	-3.67899	-0.12206	0.61628	0.20975	0.20167
-7.05865	3.16208	-0.12206	-0.61628	0.20975	-0.20167
-7.05865	-3.16208	-0.09630	0.59228	0.21512	0.19130
-5.74103	2.77609	-0.09630	-0.59228	0.21512	-0.19130
-5.74103	-2.77609	-0.07119	0.57082	0.21869	0.18045
-4.74529	2.47590	-0.07119	-0.57082	0.21869	-0.18045
-4.74529	-2.47590	-0.04841	0.55041	0.22288	0.16913
-3.97216	2.47590	-0.04841	-0.55041	0.22288	-0.16913
-3.97216	-2.23576	-0.02511	0.52987	0.22491	0.15862
-3.36257	2.04017	-0.02511	-0.52987	0.22491	-0.15862
-3.36257	-2.04017	-0.00614	0.50993	0.22840	0.14869
-2.86968	1.87670	-0.00614	-0.50993	0.22840	-0.14869
-2.86968	-1.87670	0.01222	0.49053	0.23084	0.13927
-2.46762	1.73665	0.01222	-0.49053	0.23084	-0.13927
-2.46762	-1.73665	0.02869	0.47272	0.23412	0.12944
-2.13296	1.61385	0.02869	-0.47272	0.23412	-0.12944
-2.13296	-1.61385	0.04480	0.45668	0.23747	0.12944
-1.85761	1.50791	0.04480	-0.45668	0.23747	-0.11905
-1.85761	-1.50791	0.06077	0.43943	0.23908	0.10917
-1.62230	1.14107	0.06077	-0.43943	0.23908	-0.10917
-1.62230	-1.14107	0.07471	0.42273	0.24025	0.09898
-1.42336	1.33389	0.07471	-0.42273	0.24025	-0.09898
-1.42336	-1.33389	0.56972	0.00000	0.24951	0.00410
-1.25055	1.25791	0.08758	0.40587	0.24951	-0.00410
-1.25055	-1.25791	0.08758	-0.40587	0.25096	0.01317
-1.10046	1.18935	0.09808	0.39055	0.25096	-0.01317
-1.10046	-1.18935	0.09808	-0.39055	0.25051	0.02301
-0.97255	1.12860	0.11000	0.37730	0.25051	-0.02301
-0.97255	-1.12860	0.11000	-0.37730	0.27938	0.03312
-0.85875	1.07222	0.12128	0.36257	0.24938	-0.03312
-0.85875	-1.07222	0.12128	-0.36257	0.24912	0.04222
-0.75967	1.02177	0.13192	0.34737	0.24912	-0.04222
-0.75967	-1.02177	0.13192	-0.34737	0.24711	0.05101
-0.66924	0.97531	0.13932	0.33334	0.24711	-0.05101
-0.66924	-0.97531	0.13932	-0.33334	0.24773	0.06057

and act on it by matrix A_d, so that the result will be

$$P_1 = \begin{pmatrix} X_1^{(1)} \\ \vdots \\ X_d^{(1)} \end{pmatrix} = \begin{pmatrix} a_{11} \cdots a_{1d} \\ \cdots\cdots\cdots \\ a_{d1} \cdots a_{dd} \end{pmatrix} \begin{pmatrix} X_1^{(0)} \\ \vdots \\ X_d^{(0)} \end{pmatrix} = AP_0 \pmod 1. \tag{8}$$

Then, operating on P_1 by A_d again, we find vector P_2,

$$P_2 = \begin{pmatrix} X_1^{(2)} \\ \vdots \\ X_d^{(2)} \end{pmatrix} = \begin{pmatrix} a_{11} \cdots a_{1d} \\ \cdots\cdots\cdots \\ a_{d1} \cdots a_{dd} \end{pmatrix} \begin{pmatrix} X_1^{(1)} \\ \vdots \\ X_d^{(1)} \end{pmatrix} = AP_1 \pmod 1, \tag{9}$$

and so on:

$$P_N = \{A\{A \cdots \{AP_0\}\} \cdots \}. \tag{10}$$

The points $P_0, P_1, ..., P_N$ form the very trajectory in the hypercube Π_d. In order to check the statistical features of the sequence (10) we have to calculate χ_s^2 and the discrepancy D_N [1], which determines the convergence of the Monte Carlo sums.

We arrange the components of $P_0, P_1, ...$ in the series

$$X_1^{(0)}, ..., X_d^{(0)}, X_1^{(1)}, ..., X_d^{(1)}, ..., X_1^{(2)}, ..., X_d^{(2)}, ... \tag{12}$$

As in the case of the multiplicative generator one now can compose *the words* of length K and represent them as the points of the Π^K hypercube. Varying K from 1 to d and beyond, we obtain points that fill the cube of the varying dimension K. Using the χ_s^2 criterion one can estimate the degree of uniformity of filling the cube Π^K for different K.

We have considered χ_s^2 when $K = 1, 2, 3, 4$ for the matrices A_d with different d. Each side of Π^K was divided into ten parts, so we have

$$\begin{aligned} \chi_{s-1}^2 &= \chi_9^2 && \text{for } \Pi^1, \\ \chi_{s-1}^2 &= \chi_{99}^2 && \text{for } \Pi^2, \\ \chi_{s-1}^2 &= \chi_{999}^2 && \text{for } \Pi^3, \\ \chi_{s-1}^2 &= \chi_{9999}^2 && \text{for } \Pi^4, \end{aligned} \tag{13}$$

where $s-1$ is the number of degrees of freedom: $s = 10^K$.

Let us write down values of χ_{s-1}^2 for different matrices (5):

$$\begin{aligned} A_2, \qquad & \chi_9^2 = 7.9 && X_1^{(0)} = \pi/7 \\ & \chi_{99}^2 = 106.6 && X_2^{(0)} = 1/\sqrt{2} \\ & \chi_{999}^2 = 34{,}046 && \\ & \chi_{9999}^2 = 5{,}769{,}697 && \end{aligned}$$

$$
\begin{array}{lll}
A_3, & \chi_9^2 = 10.17 & X_1^{(0)} = \pi/7 \\
 & \chi_{99}^2 = 101.3 & X_2^{(0)} = 1/\sqrt{2} \\
 & \chi_{999}^2 = 1080.8 & X_3^{(0)} = 1/\sqrt{5} \\
 & \chi_{9999}^2 = 191{,}626 & \\
A_4, & \chi_9^2 = 10.4 & X_1^{(0)} = \pi/7 \\
 & \chi_{99}^2 = 117.1 & X^{(0)} = 1/\sqrt{2} \\
 & \chi_{999}^2 = 1031 & X_3^{(0)} = 1/\sqrt{5} \\
 & \chi_{9999}^2 = 10{,}209.1 & X_4^{(0)} = 0,\ 64{,}873{,}916 \\
A_6, & \chi_9^2 = 4 & X_1^{(0)} = \pi/7 \\
 & \chi_{99}^2 = 92.3 & X_2^{(0)} = 1/\sqrt{2} \\
 & \chi_{999}^2 = 961.7 & X_3^{(0)} = X_4^{(0)} = X_5^{(0)} = X_6^{(0)} = 0 \\
 & \chi_{9999}^2 = 9848 &
\end{array}
$$

$$
\begin{array}{llll}
A_{12}, & \chi_9^2 = 9.5 & X_1^{(0)} = \pi/7 & X_8^{(0)} = \{\sqrt{13}\} \\
 & \chi_{99}^2 = 80.5 & X_2^{(0)} = 1/\sqrt{2} & X_9^{(0)} = \{\sqrt{17}\} \\
 & \chi_{999}^2 = 951.8 & X_3^{(0)} = \{\sqrt{5}\} & X_{10}^{(0)} = \{\sqrt{29}\} \\
 & \chi_{9999}^2 = 9900.8 & X_4^{(0)} = \{\sqrt{11}\} & X_{11}^{(0)} = \{\sqrt{3}\} \\
 & & X_5^{(0)} = \{\sqrt{3}\} & X_{12}^{(0)} = (1\ \sqrt{11}\} \\
 & & X_6^{(0)} = \{\sqrt{7}\} & \\
 & & X_7^{(0)} = \{\sqrt{23}\} &
\end{array}
$$

In all those examples $N/s = 24$, where N is the full amount of points and $s-1$ is the number of degrees of freedom. For comparison we give below the values of χ_{s-1}^2 for the standard multiplicative generator RNDM,

$$
\begin{aligned}
\chi_9^2 &= 8.9 \\
\chi_{99}^2 &= 97.8 \\
\chi_{999}^2 &= 1007.5 \\
\chi_{9999}^2 &= 10{,}301.1.
\end{aligned}
$$

It is seen that the statistical properties become better with the growth of the dimension of matrix A_d. Also we have calculated the dependence of χ_{s-1}^2 on the total number of generated points for the matrix A_{12}.

A_{12}

N/s	48	72	96	144
χ_9^2	7.9	10.1	6.3	9.83
χ_{99}^2	82	105.3	91.8	95
χ_{999}^2	1021	968.5	1006.9	999.5
χ_{9999}^2	9955.3	9976.4	10,069	9846

RNDM

N/s	48	72	96	144
χ_9^2	5.5	4.1	6.3	12.2
χ_{99}^2	112.3	113.9	108.4	101
χ_{999}^2	1075.7	1054	1038.8	1024
χ_{9999}^2	10321	10130.9	10249.6	10373

One can also calculate mean value $\bar{\chi}_{s-1}^2$,

$$\bar{\chi}_{s-1}^2 = \frac{\sum \chi^2(i) \cdot N_i/s-1}{\sum_i N_i/s-1}$$

for A_{12} and RNDM,

A_{12}	RNDM
9.41	9.1
90.9	104.8
1001.4	1035.2
9935.6	10321.5

As we mentioned before, the convergence of the Monte Carlo sums is connected with the behavior of the discrepance D_N. Below we give the RNDM and matrix generator A_{100} when $K=4$:

RNDM

$N \cdot 10^5$	1	2	3	4	5	6	7	8	9	10
$D_N \sqrt{N}$	1.714	1.930	1.944	2.071	2.355	2.610	2.937	3.164	3.204	3.302
χ_{9999}^2	10016	10081	10455	10202	10280	10197	10139	10199	10269	10226

A_{100}

$N \cdot 10^5$	1	2	3	4	5	6	7	8	9	10
$D_N \sqrt{N}$	1.233	1.350	1.287	1.141	1.264	1.587	1.470	1.410	1.228	1.102
χ_{9999}^2	10049	10086	9870	9751	9748	9858	9857	9901	9864	9952

All this shows that those matrix generators (we call them "mixmax") have good statistical properties and are usable for Monte Carlo simulation. The great advantage of the matrix generator is the possibility to change widely the components of the initial vector $P_0 = X_1^{(0)}, ..., X_d^{(0)}$ and matrix A_d in order to have the dependence of the Monte Carlo calculations on these parameters, which, naturally, must be weak. The proposed matrix generator can be successfully realized on computers with a matrix processor.

Reference

1. G. K. Savvidy and N. G. Ter-Arutyunyan-Savvidy, On the Monte Carlo Simulation of Physical Systems, TPI-MINN-88/2T, December 1988; Preprint EPI-865-16-86, Jan. 1986 (unpublished) [Russian]; *J. Comput. Phys.* **97**, 566 (1991).

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